2.5 Complex Zeros & the Fundamental Theorem of Algebra

Reminder: $i^2 = -1$

Exploring Fundamental Polynomial Connections

Write polynomial in standard form, & identify the zeros of the function & the x-intercepts of its graph.

ic1 $f(x) = (x-2i)(x+2i)$ ← conjugates (same in mid. opp side b/t)

$f(x) = x^2 + 2ix - 2ix - 4i^2$

$f(x) = x^2 - 4(-1)$

$f(x) = x^2 + 4$

2 zeros: $x = 2i$ & $x = -2i$

x-int: None (the zeros are not real #'s) ie2 $f(x) = (x-5)(x-\sqrt{2}i)(x+\sqrt{2}i)$

$f(x) = (x-5)(x^2 - \sqrt{4}i^2)$

$f(x) = (x-5)(x^2+2)$

$f(x) = x^3 + 2x - 5x^2 - 10$

$$\boxed{f(x) = x^3 - 5x^2 + 2x - 10}$$

3 zeros: $5, +\sqrt{2}i, -\sqrt{2}i$
(in top)

x-int: $(5,0)$

ie3 $f(x) = (x-3)(x-3)(x-i)(x+i)$

$= (x^2-6x+9)(x^2+1)$

$$\boxed{\begin{array}{l} = x^4-6x^3+10x^2-6x+9 \\ \underline{\text{Zeros}}:\ 3, 3, i, -i \\ x\text{-int}:\ (3,0) \end{array}}$$

<u>Complex Conjugates:</u> Finding a Polynomial from Given <u>Zeros</u>

(ie1) Write a Polynomial Function of minimum degree in Standard form with real coefficients whose zeros include $-3, 4, : 2-i$

* When i included it is usually 2 solutions

* Hint use opposites for solution(s)

$$(x+3)(x-4)(x-(2+i))(x-(2-i))$$

$$\uparrow \quad -3 \qquad \uparrow \quad 4 \qquad \uparrow \quad 2-i \qquad \uparrow \quad 2-i$$

$$f(x) = (x+3)(x-4)(x-(2+i))(x-(2-i))$$

$$f(x) = (x^2-x-12)(x^2-4x+5)$$

$$\boxed{= x^4-5x^3-3x^2+43x-60}$$

Finding a Polynomial from Given Zeros

(ie1) Write a polynomial function of minimum degree in standard form with real coefficients whose zeros includ $x = 1,\ 1+2i,\ 1-i$

$$(x-1)\big(x-(1-2i)\big)\big(x-(1+2i)\big)$$

1      $1+2i$

$$\big(x-(1-i)\big)\big(x-(1+i)\big)$$

$$=(x-1)(x^2-2x+5)(x^2-2x+2)$$

$$=(x^3-3x^2+7x-5)(x^2-2x+2)$$

$$\boxed{=x^5-5x^4+15x^3-25x^2+24x-10}$$

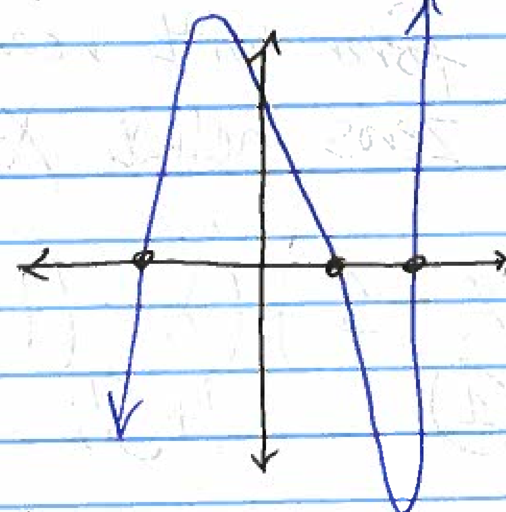
Factoring a Polynomial w/ Complex Zeros

(ex1) Find all zeros of

$$f(x) = x^5 - 3x^4 - 5x^3 + 5x^2 - 6x + 8$$

① graph it!

* 5 total zeros



Visually see 3

② Use an x-int value for synthetic division

$x = 1 ; -2 ; 4$

← missing no term

$$\begin{array}{r|rrrrrr} 1 & 1 & -3 & -5 & 5 & -6 & 8 \\ & \downarrow & 1 & -2 & -7 & -2 & -8 \\ \hline & 1 & -2 & -7 & -2 & -8 & 0 \end{array}$$

③ Or Write what is left then use -2

$$x^4 - 2x^3 - 7x^2 - 2x - 8$$

$$\begin{array}{r|rrrrr} -2 & 1 & -2 & -7 & -2 & -8 \\ & \downarrow & -2 & 8 & -2 & 8 \\ \hline & 1 & -4 & 1 & -4 & 0 \end{array}$$

④ write what is left then use 4

$$x^3 - 4x^2 + 1 - 4$$

$$\begin{array}{r|rrrr} 4 & 1 & -4 & 1 & -4 \\ & \downarrow & 4 & 0 & 4 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

⑤ (x^2+1)

last factor

$A: (x-1)(x+2)(x-4)(x^2+1)$

1   -2 4 $(x-i)(x+i)$

$A: (x-1)(x+2)(x-4)(x-i)(x+i)$

Final Answer simplified

Finding Complex Zeros

ie 1 $Z = 1-2i$ a zero of ~~f(x)~~

$f(x) = 4x^4 + 17x^2 + 14x + 65$

missing x^3 ☺

① * if $(1-2i)$ is a factor so is... $(1+2i)$

② Sythetic division using factors

$$
\begin{array}{r|rrrrr}
1-2i & 4 & 0 & 17 & 14 & 65 \\
 & \downarrow & 4-8i & -12-16i & -27+26i & -65 \\
\hline
 & 4 & 4-8i & 5-16i & -13-26i & ☺
\end{array}
$$

$(1-2i)(4-8i)$

$4-8i-8i+16(i^2)$

$4-16i-16$

$-12-16i$

$(1-2i)(5-16i)$

$5-16i-10i+32i^2$

$5-26i-32$

$-27-26i$

$(1-2i)(-13-26i)$

$-13-\cancel{26i}+\cancel{26i}+52i^2$

$-13+52(-1)$

-65

③ do it again with $1+2i$

$4x^3 + (4-8i) + (5-16i) + (-13-26i)$

$$\begin{array}{c|cccc} 1+2i & 4 & 4-8i & 5-16i & -13-26i \\ & \downarrow & 4+8i & 8+16i & 13+26i \\ \hline & 4 & 8 & 13 & 0 \end{array}$$

(4) Write the function

$f(x) = 4x^2 + 8x + 13$

$(\quad)(\quad)$ a) not factorable

b) then graph . . . no help

c) quad formula

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$a = 4$
$b = 8$
$c = 13$

$$x = \frac{-8}{2(4)} \pm \frac{\sqrt{8^2-4(4)(13)}}{2(4)}$$

$$x = -1 \pm \frac{\sqrt{64-208}}{8}$$

$$x = -1 \pm \frac{\sqrt{-144}}{8} \rightarrow -1 \pm \frac{i\sqrt{144}}{8}$$

$$-1 \pm \frac{\cancel{12}^{4} i}{\cancel{8}_{4}} \Rightarrow -1 \pm \frac{3i}{2}$$

Write the zeros

$$(1-2i)(1+2i)\left(-1+\frac{3i}{2}\right)\left(-1-\frac{3i}{2}\right)$$

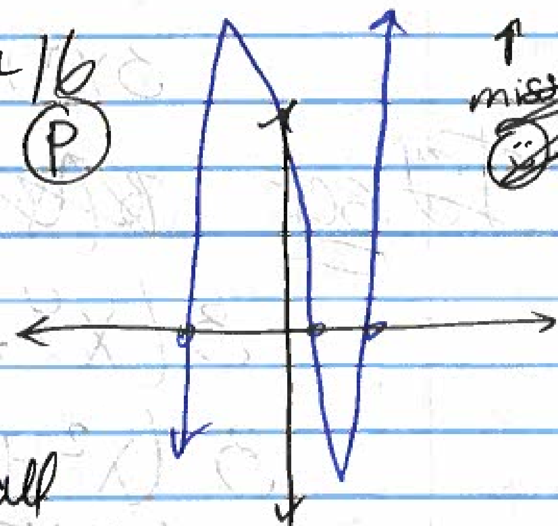
Write as a function:

$$f(x) = (x-(1-2i))(x-(1+2i)) \cdot \left(x-\left(-1+\frac{3i}{2}\right)\right)\left(x-\left(-1-\frac{3}{2}i\right)\right)$$

Factoring a Polynomial:

ie 1 $f(x) = 3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16$

Q P



miss ~~x~~

① Graph it! to find x-int.

$(-1.414, 0)$

$(.666, 0)$ 3 all approx.

$(1.414, 0)$

3 zeros

no good for us

② try $\frac{p}{q}$ method

$$\frac{16}{3} = \frac{(\pm 1, \pm 2, \pm 4, \pm 8, \pm 16)}{\pm 1, \pm 3}$$

all decimal so try a fraction

$\frac{18}{3} - \frac{1}{3} = \frac{17}{3}$ $-\frac{36}{9} + \frac{17}{9}$

nu $\frac{1}{3}$	3	-2	6	-4	-24	$+16$
	↓	1	$-\frac{1}{3}$	$\frac{17}{9}$	$\frac{19}{27}$	
	3	-1	$\frac{17}{3}$	$\frac{19}{39}$	$\frac{19}{27}$	not zer

is 0.666

Try $\frac{2}{3}$

3	-2	6	-4	-24	16
↓	2	0	4	0	-16
3	0	6	☺	-24	☺

yet

$$3x^4 + 0x^3 + 6x^2 + 0x - 24$$

Try GCF $\cancel{3(x^4 + 0x^3 + 2x^2)}$

$$3(x^4 + 2x^2 - 8)$$

$$3(x^2 + 4)(x^2 + {}^-2)$$

$$\cancel{3(\quad)(\quad)}$$

$\cancel{(x - \frac{2}{3})}$ $+\frac{2}{3}$ $\frac{2}{3}$

$$3(x^2 + 4)(x + \sqrt{2})(x - \sqrt{2})$$

~~3 gets changed due to GCF~~

$$A: (x - \tfrac{2}{3})(x + \sqrt{2})(x - \sqrt{2})(x^2 + 4)$$

2/3

using 3

$$(3x - 2)(x + \sqrt{2})(x - \sqrt{2})(x + \sqrt{2})$$

2/3

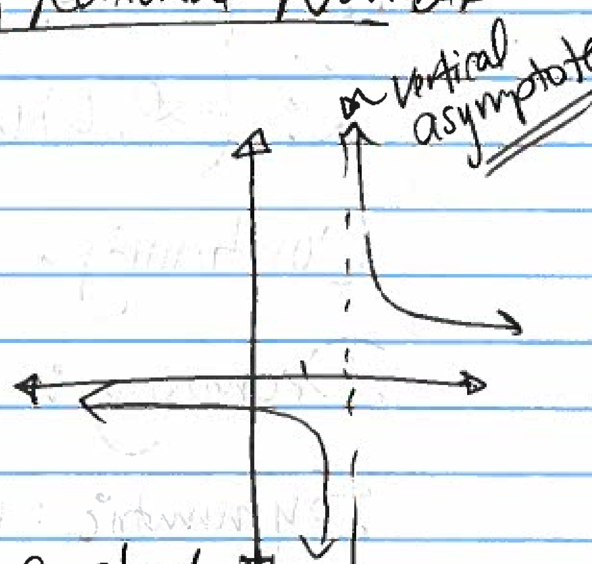
* Every Polynomial function with odd degree has at least 1 real zero

2.6 Graphs of Rational Functions

Rational Function: $r(x) = \frac{f(x)}{g(x)}$

Finding the Domain of Rational Numbers

ie1 $f(x) = \frac{1}{x-2}$



$\lim_{x \to 2^-} f(x) = -\infty$

End Behavior

as x approaches 2 from the negative side

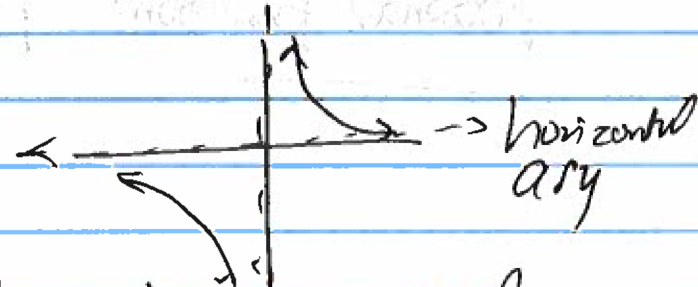
D: $(-\infty, 2) \cup (2, \infty)$

$\lim_{x \to 2^+} f(x) = \infty$

as x approaches 2 from the positive side

Transformations of the Reciprocal Function

$f(x) = \frac{1}{x}$



D: $(-\infty, 0) \cup (0, \infty)$

R: $(-\infty, 0) \cup (0, \infty)$

Continuity: All $x \neq 0$

Decreasing: $(-\infty, 0)$; $(0, \infty)$

Symmetric: WRT origin

Unbounded

No Local Extrema

Horizontal Asymptote: $y = 0$

Vertical Asymptote: $x = 0$

End Behaviors $\lim_{x \to -\infty} = 0$ $\quad \lim_{x \to +\infty} = 0$

Transforming the Reciprocal Functions

ie 1 $g(x) = \dfrac{2}{x+3}$

Write as
$f(x) =$
$g(x) =$

parent $= f(x) = \dfrac{1}{x}$

$f(x) = \left(\dfrac{1}{x+3}\right)$ $\qquad g(x) = 2\left(\dfrac{1}{x+3}\right)$

$= 2f(x+3)$

$\boxed{\text{V.A } x=-3}$

$\boxed{\text{HA } y=0}$

D: $(-\infty,-3)\cup(-3,\infty)$

R: $(-\infty,0)\cup(0,\infty)$

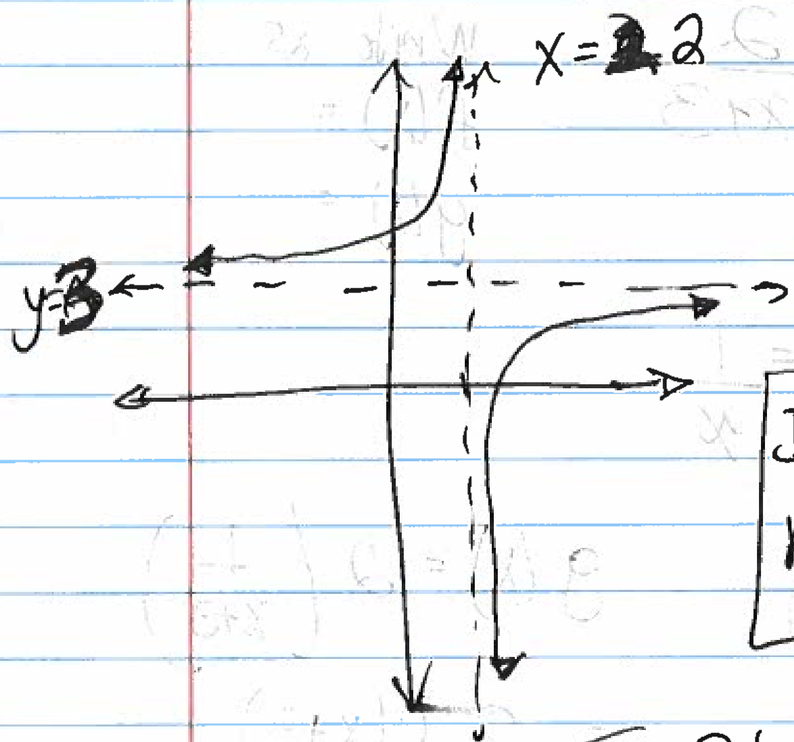
E.B

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = +\infty$

(3)

ie2 $h(x) = \dfrac{3x-7}{x-2}$



$\boxed{\begin{array}{l} VA: x = 2 \\ HA: y = 3 \end{array}}$

$\boxed{\begin{array}{l} D: (-\infty, 2) \cup (2, \infty) \\ R: (-\infty, 3) \cup (3, \infty) \end{array}}$

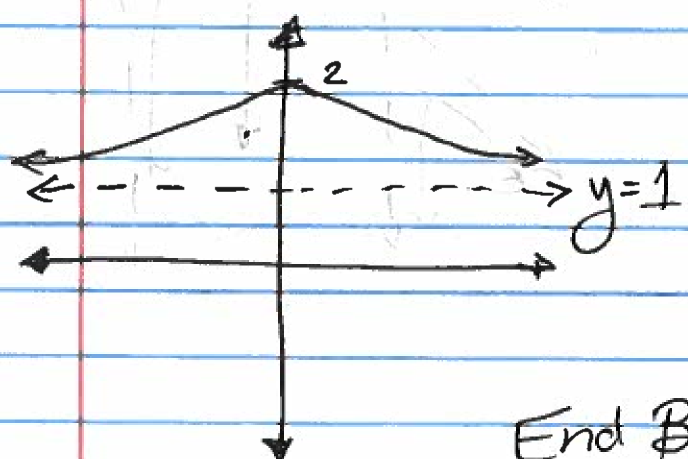
End Behaviors:

$\lim\limits_{x \to 1^-} f(x) = -\infty$ $\qquad$ $\lim\limits_{x \to 1^+} f(x) = +\infty$

Limits & Asymptotes:

Finding Asymptotes:

ie1 $f(x) = \frac{(x^2+2)}{(x^2+1)}$



HA: $y=1$

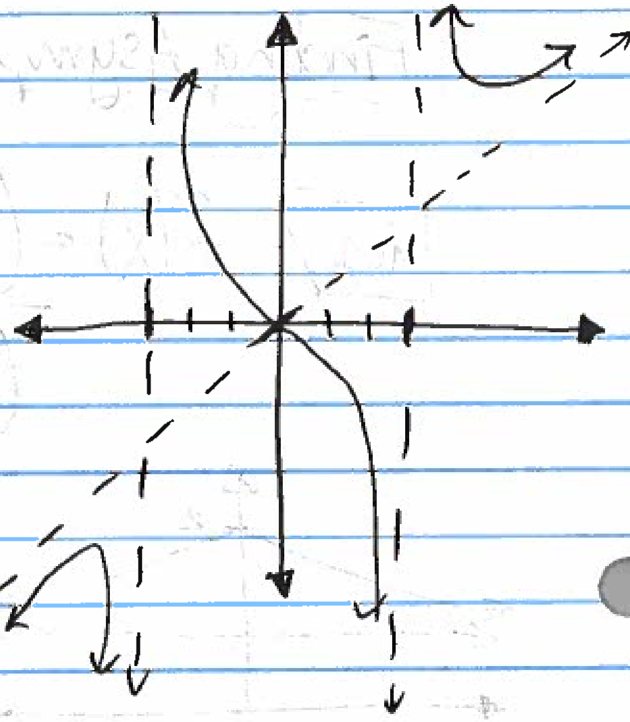
VA: none

End Behaviors:

$\lim_{x \to -\infty} f(x) = 1$ $\lim_{x \to +\infty} f(x) = 1$

Slant Asymptotes: an asymptote with a slant line

ie1 $f(x) = \frac{x^3}{x^2-9}$



HA: $x=3$
$x=-3$

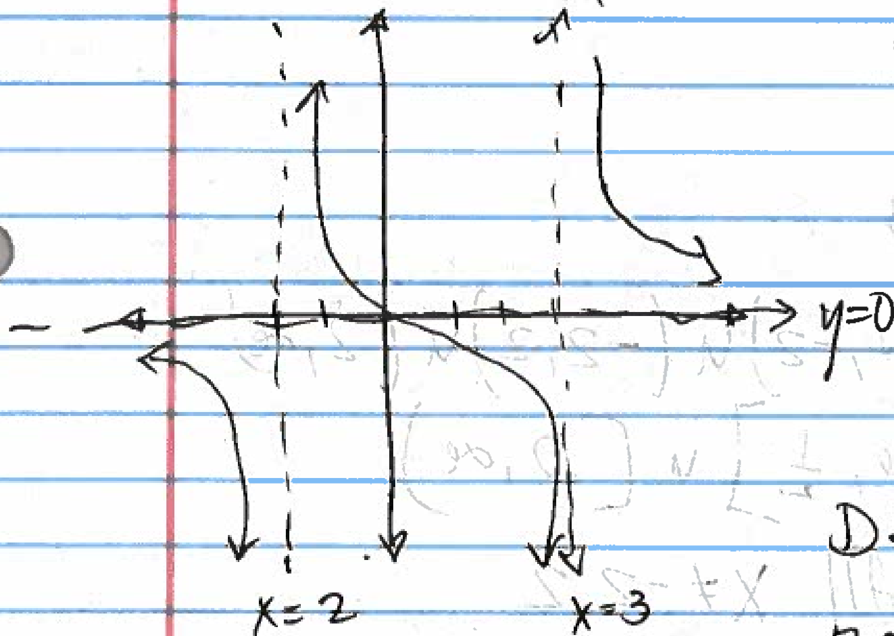
VA:
↓ or

Slant asympt: $y=x$

Analyzing Graphs of Rational Functions

ie1 $f(x) = \dfrac{x-1}{x^2-x-6}$

$(x-3)(x+2)$



~~HA~~:

VA: $x=-2$

$x=3$

HA: $y=0$

D: $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$

R: $(-\infty,\infty)$

E.B.

$\lim_{x\to-\infty} f(x)=0$

$\lim_{x\to+\infty} f(y)=0$

Continuous: All $x \neq -2$ & 3

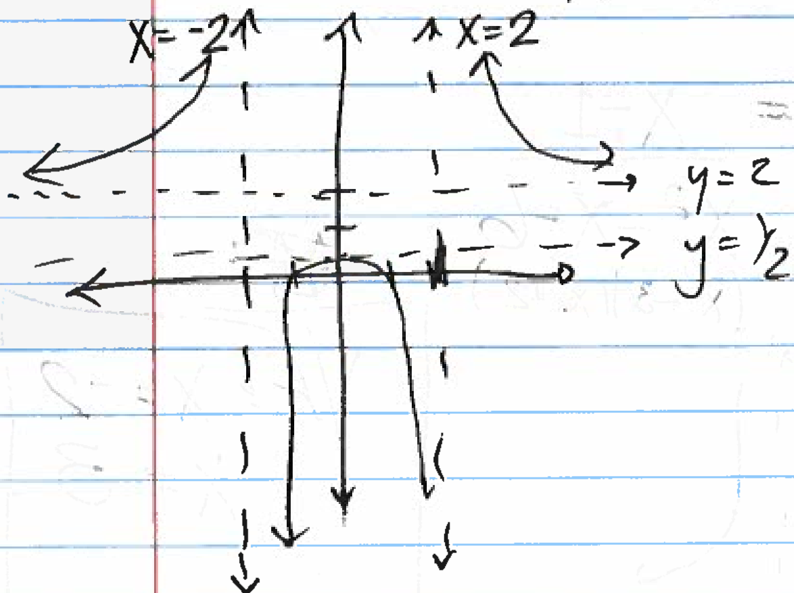
Decreasing: $(-\infty,-2)$, $(-2,3)$ & $(3,\infty)$

Not Symmetric

Unbounded

No local extremes

$\boxed{\text{ie } 2}$ $f(x) = \dfrac{2x^2-2}{x^2-4}$



D: $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

R: $(-\infty, \frac{1}{2}] \cup [2,\infty)$

Continuity: All $x \neq -2, 2$

Increasing on $(-\infty,-2)$; $(-2,0]$

Decreasing on $[0,2)$; $(2,\infty)$

Symmetric WRT y-axis (even function)

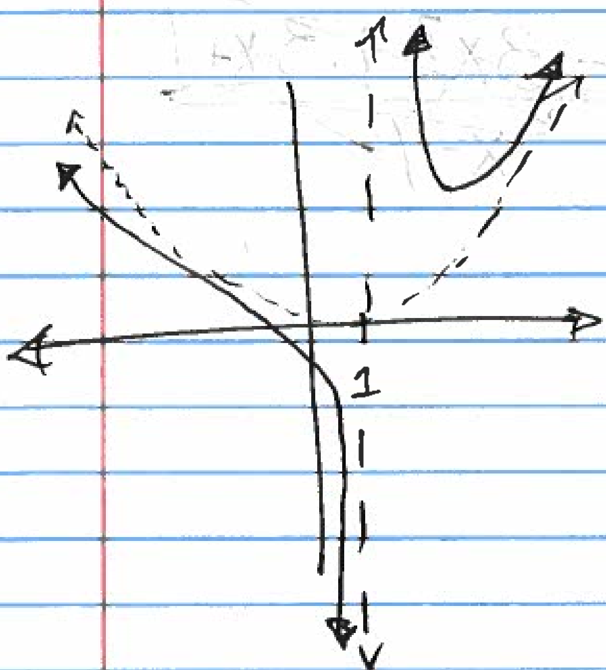
HA: $y=2$; $y=\frac{1}{2}$

VA: $x=-2$; $x=2$

E.B. $\lim_{x\to-\infty} f(x) = 2$ $\lim_{x\to+\infty} f(x) = 2$

Finding End-Behavior Asymptote

ie1 $f(x) = \dfrac{x^3-3x^2+3x+1}{x-1}$



E.B.

$\lim_{x \to -\infty} f(x) = \infty$

$\lim_{x \to +\infty} f(x) = \infty$

Domain: $(-\infty, 1) \cup (1, +\infty)$

R: $(\infty, +\infty)$ ℝ

~~NO~~

HA: none

VA: $x=1$

Continuous: All $x \neq 1$

Decreasing: $(-\infty, 1)$ and $(1, 2]$

Increasing: $[2, \infty)$

Not Symmetric

Unbounded

Local min ~~of~~ 3 @ $x=2$ (9)

~~Analyzing the Graph of a Rational Function:~~

ex 1 $f(x) = \dfrac{x^3 - 3x^2 + 3x + 1}{x - 1}$

2.7 Solution Equations in 1 Variable

Extraneous Solutions: an answer that does NOT meet the requirements of the solution.

Least Common Denominator: for all terms in the denominators and can include variable.

Solving by Clearing Fractions

ie 1 $x + \frac{3}{x} = 4$

$$\left[\frac{x}{1} + \frac{3}{x} = \frac{4}{1}\right] x$$

~~Kebm~~ = 1 LCD

$$x^2 + 3 = 4x$$ subtract $4x$

$$-4x \quad -4x$$

$$x^2 - 4x + 3 = ☺$$ factor for x-int

$$(x-3)(x-1) = ☺$$

$$x = 3 \quad x = 1$$

1

Now Look for extraneous Solutions
back into original

$$x + \frac{3}{x} = 4$$

1 $\quad 1 + \frac{3}{1} = 4$

yes $\quad 1 + 3 = 4$

3 $\quad 3 + \frac{3}{3} = 4$

yes $\quad 3 + 1 = 4$

When $x = 1$
& $x = 3$

Both check out

No extraneous solutions

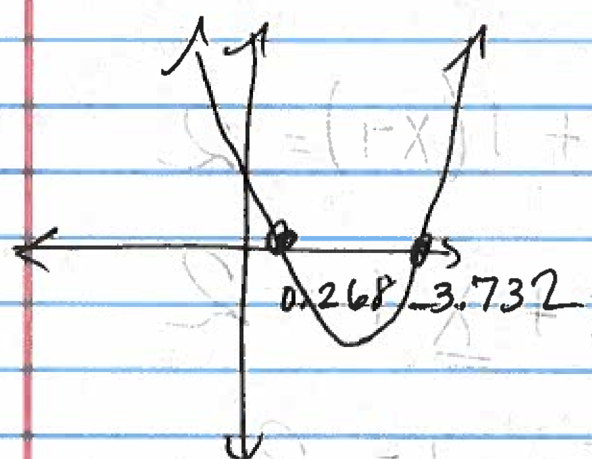
ic2 $x + \frac{1}{x-4} = 0$

$x-4\left(\frac{x}{1} + \frac{1}{x-4}\right) = 0$ LCD $= (x-4)$

$(x-4)(x) + 1 = 0$

$x^2 - 4x + 1 = 0$

1) graph it
2) factor (no)



$x = 0.268$

$x = 3.732$

check both ch

0.268

$\frac{0.268}{1} + \frac{1}{0.268-4} = 0$

$\frac{0.268}{1} + \frac{-0.268}{1} = 0$ ✓

$0 = 0$

$\frac{3.732}{1} + \frac{1}{3.732-4} = 0$

$\frac{3.732}{1} + \frac{1}{-0.268} = 0$

$3.732 + -3.732 = 0$

$0 = 0$ (3)

Extraneous Solutions

ie 3

$$\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{x^2-4x+3}$$

① ALL must in smallest X degree

$$(x-1)(x-3)\left[\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{(x-3)(x-1)}\right]$$

② Find LCD

$(x-1)(x-3)$

$$2x(x-3) + 1(x-1) = 2$$

③ Solve w/ LCD denom. with be eliminate

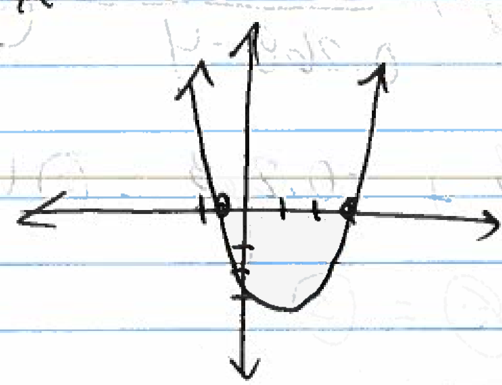
$$2x^2 - \underline{6x} + \underline{x} - 1 = 2$$

$$2x^2 - 5x - 1 = 2$$

$$\quad -2 \quad -2$$

$$2x^2 - 5x - 3 = 0$$

④ graph it! or factor



$x = -0.5$

$x = 3$

⑤ check for extraneous solutions

$x = -0.5$

$x = 3$

test -0.5

$$\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{(x^2-4x+3}$$

$$\frac{2\left(-\frac{1}{2}\right)}{-0.5-1} + \frac{1}{-0.5-3} = \frac{2}{-0.5^2-4(-0.5)+3)}$$

$$7 \cdot \frac{2}{3} - \frac{2 \cdot 3}{7} = \frac{8}{21}$$

$$\frac{14}{21} - \frac{6}{21} = \frac{8}{21}$$

$$\frac{8}{21} = \frac{8}{21}$$

$\boxed{x = -0.5}$ good solution

Test 3

$$\frac{2x}{x-1}+\frac{1}{x-3}=\frac{2}{x^2-4x+3}$$

$$\frac{2(3)}{3-1}+\frac{1}{3-3}$$

$$\frac{6}{2}+\frac{1}{0}$$

Not possible

$$3$$

So $x=3$ is extraneous an Solution

(ie2) $$\frac{x-3}{x} + \frac{3}{x+2} + \frac{6}{x^2+2x} = ☺$$

① all smallest degrees

$$\frac{x-3}{x} + \frac{3}{x+2} + \frac{6}{x(x+2)} = ☺$$

② Find LCD

Remove Denominator

$$x(x+2)\left[\frac{x-3}{x} + \frac{3}{x+2} + \frac{6}{x(x+2)}\right] = 0$$

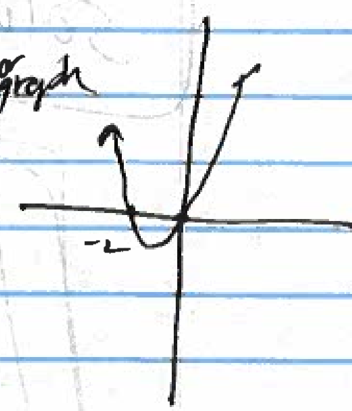
$$(x+2)(x-3) + 3(x) + 6 = 0$$

③ Simplify to graph or factor

$$x^2 - 1x - 6 + 3x + 6 = 0$$

$$x^2 + 2x = 0 \quad \text{graph}$$

$$x(x+2) = 0 \quad \text{factor}$$



④ Find x-intercepts

$x = ☺$ $\quad x = -2$

(5) Check solutions

$x = 0$ $\quad$ $x = -2$

$\boxed{0}$ $\quad$ $\boxed{-2}$

$$\frac{x-3}{x} + \frac{3}{x+2} = \frac{6}{x^2+2x} = 0$$

$$\frac{3}{-2-2} = \frac{3}{0}$$

$$\frac{0-3}{0}$$

$\rightarrow 0$

Not valid extraneous solution

not valid extraneous solution

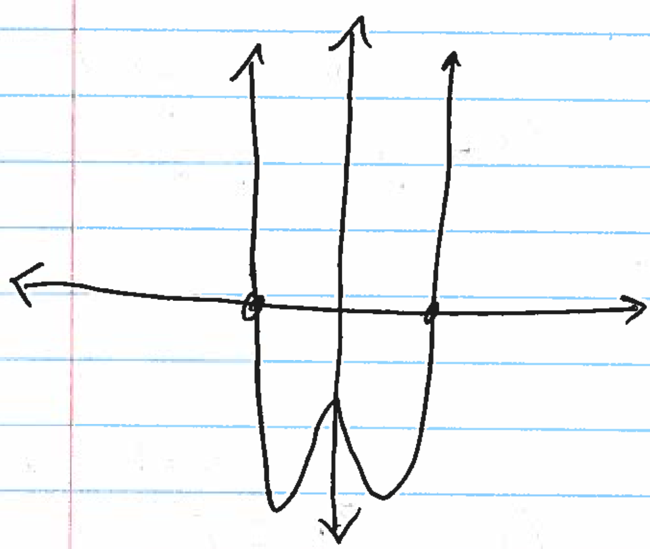
Neither Solution works! in original solutions

2.8. Solving Inequalities in 1 Variable

Finding Where a Polynomial is Zero, Positive, or Negative

ie1 $f(x) = (x+3)(x^2+1)(x-4)^2$

when $f(x) =$
a) zero $x = -3$, $x = 4$
b) positive x^2+1 all positive #'s, $x = 4$
c) negative $x = -3$



Dilton,

- The solution

$$(x+3)(x^2+1)(x-4)^2 > 0$$ positive

is $(-3,4) \cup (4,\infty)$

- The solution

$$(x+3)(x^2+1)(x-4)^2 \geq 0$$ zero & positive

is $[-3,\infty)$

- The solution

$$(x+3)(x^2+1)(x-4)^2 < 0$$ negative

is $(-\infty,-3)$

- The solution

$$(x+3)(x^2+1)(x-4)^2 \leq 0$$ zero & negative

is $(-\infty,-3] \cup \{4\}$

Solving a Polynomial Inequality Analytically

① graph
② Synth division

ie 1 $2x^3 - 7x^2 - 10x + 24 > 0$ ← no term missing

$$\begin{array}{r|rrrr} 4 & 2 & -7 & -10 & 24 \\ & + & 8 & 4 & -24 \\ \hline & 2 & 1 & -6 & ☺ \end{array}$$

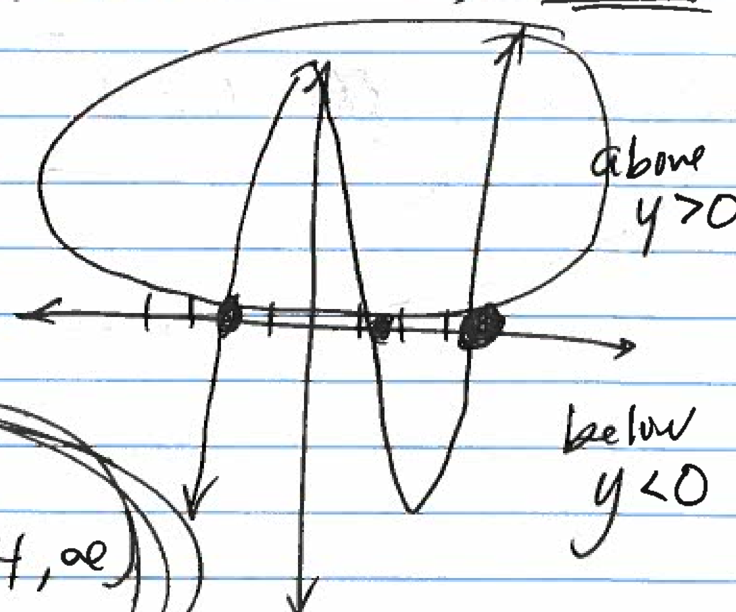
$2x^2 + 1x - 6 \geq$ ☺

$(x-4)(2x-3)(1x+2) \geq$ ☺

$x = 4$

$x = \frac{3}{2}$

$x = -2$



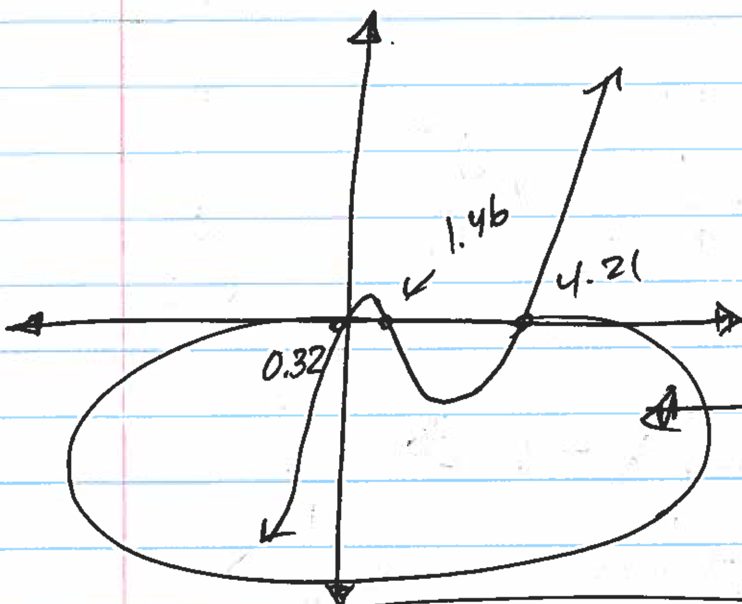
is

$(-2, \frac{3}{2}) \cup (4, \infty)$

$\boxed{\text{ie2}}$

$$\begin{aligned} x^3 - 6x^2 &\le 2 - 8x \\ +8x \quad -2 &\quad -2 + 8x \end{aligned}$$

$$x^3 - 6x^2 + 8x - 2 \le ☺$$



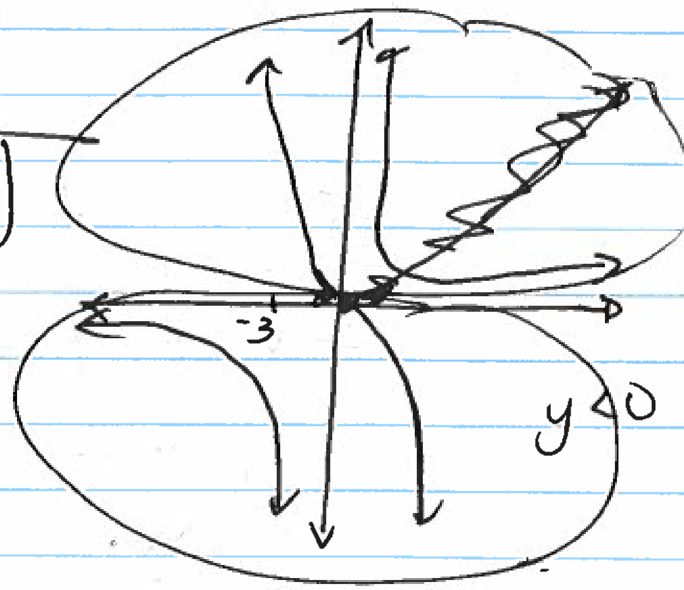
$y \le 0$

included by = sign

is $(-\infty, 0.32] \cup [1.46, 4.21]$

Rational Inequalities:

$$r(x) = \frac{(2x+1)}{((x+3)(x-1))}$$



$x = -\frac{1}{2}$

$x \neq 1$ $x \neq -3$

$y > 0$ is $(-3, -\frac{1}{2}) \cup (1, \infty)$

$y = 0$ is undefined

$x \neq 1$ $x = -3$

$y < 0$ ~~is none below y-line (axis)~~

is $(-\infty, -3) \cup (-\frac{1}{2}, 1)$

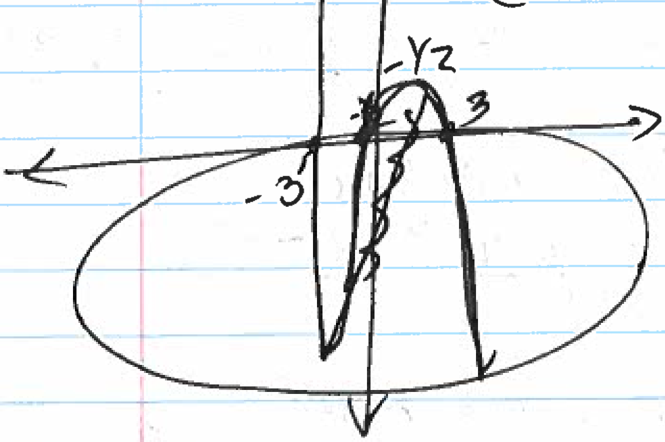
ie2

$$\left(\frac{5}{x+3} + \frac{3}{x-1} < \text{☺}\right)(x+3)(x-1)$$

$$5(x-1) + 3(x+3) < \text{☺}$$

$$5x - 5 + 3x + 9 < \text{☺}$$

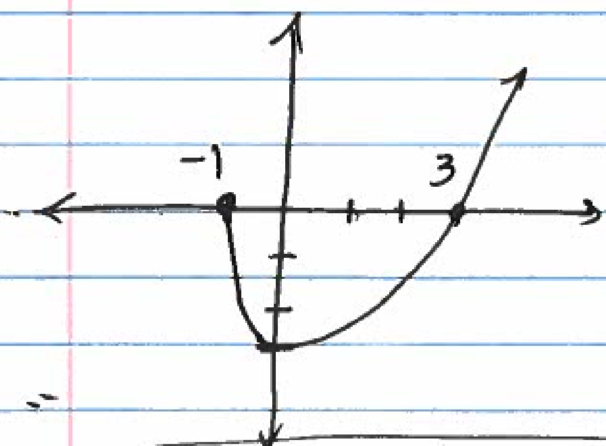
$$\frac{8x+4 < \text{☺}}{(x+3)(x-1)}$$



$y<0$ is $(-\infty, -3) \cup (-\frac{1}{2}, 1)$

Other Inequalities:

$\boxed{\text{ie 1}}$ $(x-3)\cdot\sqrt{x+1} \geq 0$



$\boxed{y \geq 0 \text{ is } \{-1\} \cup [3,\infty)}$